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NORMALIZING CONSTANTS FOR BRANCHING PROCESSES IN RANDOM ENVIRONMENTS (B.P.R.E.)*

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Normalizing constants are obtained for B.P.R.E. such that the limiting random variable is finite almost everywhere and is zero only on the extinction set of the process w.p.1. Furthermore, the normalizing constants can be chosen so that they grow exponentially fast, and so that the ratio of successive constants converges in distribution. The method of proof used is to prove the result for increasing branching processes, and then, to transfer the result to general B.P.R.E. by employing the relationships between B.P.R.E., the associated B.P.R.E., and the reduced branching process.

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1. Introduction

Let $\{Z_n\}_{n=0}^{\infty}$ be a one-dimensional Galton–Watson process (see [4]). Senata [7] showed that there exists a sequence of normalizing constants $\{c_n\}_{n=0}^{\infty}$ such that $c_n^{-1}Z_n$ converges in distribution to a positive random variable W which is zero only on the extinction set of the process. In 1970, Heyde, using an exponential martingale, strengthened the result to almost everywhere convergence. In [8], Senata gives a shorter proof of the almost everywhere convergence and, in addition, finds normalizing constants for "branching processes in a varying environment" (see [6]). Unfortunately, his limit random variable W may be infinite on some set having positive probability and, furthermore, may be zero on a set strictly larger than the extinction set of the process.

In this paper, we seek the analogue for B.P.R.E. (see [2] for the definition of B.P.R.E.) of the normalization theorem proved for Galton–Watson processes. In particular, we prove the following theorem.

Theorem 1. *Let $\{Z_n\}_{n=0}^{\infty}$ be a B.P.R.E. with a stationary and ergodic environmental sequence $\bar{\xi}$ such that $\mathbf{E}|\log m(\xi_0)| < \infty$ (where $m(\xi_0)$ is the expected number of offspring of a particle conditioned on the environment ξ_0). Then there exists a sequence*

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of random variables $c_n(\bar{\xi})$, depending only on the environmental sequence $\bar{\xi}$, with the following properties:

(1) $\lim_{n\to\infty} c_n^{-1} Z_n = W$ *w.p.*1,

(2) $\mathbf{P}(W = 0 \mid \bar{\xi}) = q(\bar{\xi})$ (= *extinction probability conditioned on* $\bar{\xi}$),

(3) $\mathbf{P}(W < \infty \mid \bar{\xi}) = 1$ *w.p.*1,

(4) $\lim_{n\to\infty} 1/n \log c_n(\bar{\xi}) = \mathbf{E}(\log m(\xi_0))$ *w.p.*1, *and*

(5) $c_{n+1}(\bar{\xi})/c_n(\bar{\xi})$ *converges in distribution to* $m(\xi_0)$ *as* $n \to \infty$.

Theorem 1 is first proved for increasing B.P.R.E.'s, namely those B.P.R.E.'s for which $\mathbf{P}(p_0(\xi_0) = 0) = 1$ where $p_0(\xi_0)$ is the probability of a particle giving birth to zero children (i.e., the particle "dies") given the environment ξ_0. Using Theorem 4.5 of [9], this leads to a normalizing result for the reduced branching process $\{W_n\}_{n=0}^{\infty}$ ([9]), the particles of the B.P.R.E. having infinite descent. Finally, in Section 3 we prove the following theorem which relates the normalizing of the reduced B.P.R.E. to that of the B.P.R.E.

Theorem 2. *Let $\{Z_n\}_{n=0}^{\infty}$ be a B.P.R.E. for which $\mathbf{E}(\log m(\xi_0)) > 0$. Let $A = \{\omega: Z_n(\omega) \to +\infty$ as $n \to \infty\}$. Then, for almost all $\bar{\xi}$,*

$$\mathbf{P}\left(A, \lim_{n\to\infty} \frac{W_n}{(1 - q(T^n\bar{\xi}))Z_n} = 1 \,\Big|\, \bar{\xi}\right) = \mathbf{P}(A \mid \bar{\xi}) = 1 - q(\bar{\xi}).$$

2. Normalizing increasing B.P.R.E.'s

Let $\{Z_n\}_{n=0}^{\infty}$ be an increasing B.P.R.E. with stationary and ergodic environmental sequence $\bar{\xi} = \{\xi_i\}_{i=0}^{\infty}$. By Theorem 6.5 of [3], we may extend the $\bar{\xi}$ process into the "past" in a stationary and ergodic manner. We will still denote this extended process by $\bar{\xi}$, *and it is this extended process which we shall continue to use throughout this paper.* Let $m(\xi_i)$ denote the expected number of offspring of a particle given the environment ξ_i. *It is assumed that* $\mathbf{E}(\log m(\xi_0))$ *exists and is finite.* Let $\{f_{\xi_i}\}_{i=-\infty}^{+\infty}$ be the sequence of probability generating functions (p.g.f.'s) associated with $\bar{\xi}$. Since $f_{\xi_i}(0) = 0$ and $f_{\xi_i}(1) = 1$ and $f_{\xi_i}(s)$ is increasing as s increases, $s \in [0, 1]$, we can define the inverse function $g_{\xi_i}(s)$ on $[0, 1]$.

Lemma 2.1. *For* $-\infty < i < +\infty$,

(1) $g_{\xi_i}(s) \uparrow$ *as* $s \uparrow$,

(2) $g_{\xi_i}(s) \geq s$,

(3) $g'_{\xi_i}(s) \downarrow$ *as* $s \uparrow$, *and*

(4) $1 \geq (1 - g_{\xi_i}(s))/(1 - s) \downarrow 1/m(\xi_i)$ *as* $s \uparrow 1$.

Proof. Parts (1), (2), and (3) are clear. As for part (4), we note that by (2) the first inequality in (4) follows. Also by the Mean Value Theorem

$$\frac{1-g_{\xi_i}(s)}{1-s} = \frac{1}{f'_{\xi_i}(g_{\xi_i}(t))} \tag{1}$$

for some $t \in (s, 1)$. Now $\lim_{t\to 1} g_{\xi_i}(t) = 1$ and $f'_{\xi_i}(1) = m(\xi_i)$, and hence

$$\lim_{s\uparrow 1} \frac{1-g_{\xi_i}(s)}{1-s} = \frac{1}{m(\xi_i)}\,. \tag{2}$$

Furthermore, we note that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1-g_{\xi_i}(s)}{1-s}\right] = \frac{-(1-s)g'_{\xi_i}(s)+(1-g_{\xi_i}(s))}{(1-s)^2}\,. \tag{3}$$

Using equation (1) with part (3) of the lemma, it readily follows that the numerator in the right-hand side of (3) is negative so that $(1-g_{\xi_i}(s))/(1-s)$ is decreasing to $1/m(\xi_i)$ by (2).

Lemma 2.2. (1) $g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s))\cdots)\uparrow 1$ *as* $n\to\infty$ *for* $s\neq 0$, *and*
(2) $g_{\xi_0}(g_{\xi_{-1}}(\cdots(g_{\xi_{-n}}(s))\cdots)\uparrow 1$ *as* $n\to\infty$ *for* $s\neq 0$.

Proof. (1) Clearly $\overline{\lim}_{n\to\infty} g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s))\cdots)\leq 1$. By part (2) of Lemma 2.1, $g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s))\cdots)$ increases as n increases. Hence, there exists $0<d\leq 1$ such that $\lim_{n\to\infty} g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s))\cdots)=d$. Suppose $d<1$. Then, for $s\neq 0$,

$$\begin{aligned} s &= f_{\xi_0}(f_{\xi_1}(\cdots(f_{\xi_n}(g_{\xi_n}(\cdots(g_{\xi_0}(s))\cdots) \\ &\leq f_{\xi_0}(f_{\xi_1}(\cdots(f_{\xi_n}(d))\cdots)<1 \end{aligned} \tag{4}$$

where the first inequality follows from part (2) of Lemma 2.1. But by Theorem 6 of [1], $\lim_{n\to\infty} f_{\xi_0}(f_{\xi_1}(\cdots(f_{\xi_n}(d))\cdots)=0$ (since the extinction probability is zero) thus contradicting (4). Therefore $d=1$ and part (1) of the lemma is proved.

(2) By part (1) of Lemma 2.1, it follows that $g_{\xi_0}(g_{\xi_{-1}}(\cdots(g_{\xi_{-n+1}}(s))\cdots)$ increases as s increases. By part (2) of Lemma 2.1, $g_{\xi_{-n}}(s)\geq s$, and increases as n increases to a limit d_1, where $0<d_1\leq 1$. But $g_{\xi_0}(g_{\xi_{-1}}(\cdots(g_{\xi_{-n}}(s))\cdots)$ equals $g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s))\cdots)$ in distribution. Hence, by part (1), we conclude that $d_1=1$.

Lemma 2.3.

$$\lim_{n\to\infty} \mathbf{E}\left(\log\frac{(1-g_{\xi_0}(g_{\xi_{-1}}(\cdots(g_{\xi_{-n}}(s))\cdots))}{(1-g_{\xi_{-1}}(g_{\xi_{-2}}(\cdots(g_{\xi_{-n}}(s))\cdots))}\right) = -\mathbf{E}(\log m(\xi_0)).$$

Proof. By L'Hospital's Rule,

$$\lim_{x\uparrow 1}\frac{1-g_{\xi_0}(x)}{1-x} = g'_{\xi_0}(1) = \frac{1}{m(\xi_0)}\,, \tag{5}$$

and since by the stationarity of $\bar{\xi}$ and part (2) of Lemma 2.2 $\lim_{n\to\infty} g_{\xi_{-1}}(g_{\xi_{-2}}(\cdots(g_{\xi_{-n}}(s))\cdots)=1$, then

$$\lim_{n\to\infty} \frac{1-g_{\xi_0}(g_{\xi_{-1}}(\cdots(g_{\xi_{-n}}(s))\cdots)}{1-g_{\xi_{-1}}(g_{\xi_{-2}}(\cdots(g_{\xi_{-n}}(s))\cdots)} = \frac{1}{m(\xi_0)}. \tag{6}$$

Hence

$$\lim_{n\to\infty} \log\left(\frac{1-g_{\xi_0}(g_{\xi_{-1}}(\cdots(g_{\xi_{-n}}(s))\cdots)}{1-g_{\xi_{-1}}(g_{\xi_{-2}}(\cdots(g_{\xi_{-n}}(s))\cdots)}\right) = -\log m(\xi_0). \tag{7}$$

Now, by part (4) of Lemma 2.1,

$$0 \leq \left|\log\left(\frac{1-g_{\xi_0}(g_{\xi_{-1}}(\cdots(g_{\xi_{-n}}(s))\cdots)}{1-g_{\xi_{-1}}(g_{\xi_{-2}}(\cdots(g_{\xi_{-n}}(s))\cdots)}\right)\right| \leq |\log m(\xi_0)| \tag{8}$$

for all n since $m(\xi_0) \geq 1$. But $\mathbf{E}|\log m(\xi_0)| < \infty$, so applying the Lebesgue Dominated Convergence Theorem yields the lemma.

Let $s_0 \in (0,1)$ be fixed. Define $a_0(\bar{\xi}) = s_0$ and $a_n(\bar{\xi}) = g_{\xi_{n-1}}(g_{\xi_{n-2}}(\cdots(g_{\xi_0}(s_0))\cdots)$ for $n \geq 1$.

Lemma 2.4. (1) $0 < \log a_{n+1}(\bar{\xi})/\log a_n(T\bar{\xi}) \leq 1$, *where T is the shift transformation, and*

(2) $\lim_{n\to\infty} \log a_{n+1}(\bar{\xi})/\log a_n(T\bar{\xi}) = d(\bar{\xi})$ *w.p.1, where* $0 < d(\bar{\xi}) \leq 1$ *w.p.1.*

Proof. (1) Since

$$1 > g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s_0))\cdots) \geq g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_1}(s_0))\cdots) > 0, \tag{9}$$

$$0 < \frac{|\log g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s_0))\cdots)|}{|\log g_{\xi_n}(g_{\xi_{n-1}}(\cdots g_{\xi_1}(s_0))\cdots)|} \leq 1. \tag{10}$$

But it is easily seen that the inner expression in (10) is just $\log a_{n+1}(\bar{\xi})/\log a_n(T\bar{\xi})$, so part (1) is proved.

(2) Since $0 < a_{n+1}(\bar{\xi}) < 1$, and $0 < a_n(T\bar{\xi}) < 1$ and since both $a_n(\bar{\xi})$ and $a_n(T\bar{\xi})$ converge to 1 as $n \to \infty$ (by part (1) of Lemma 2.2 and the invariance of $\bar{\xi}$ under the shift T), it follows that

$$\lim_{n\to\infty} \frac{\log a_{n+1}(\bar{\xi})}{\log a_n(T\bar{\xi})} \cdot \frac{(1-a_n(T\bar{\xi}))}{(1-a_{n+1}(\bar{\xi}))} = 1. \tag{11}$$

(Note that (11) is easily verified by using the Taylor series expansion for $\log(1+x)$, $|x| < 1$.) Now

$$\frac{1-a_{n+1}(\bar{\xi})}{1-a_n(T\bar{\xi})} = \frac{\dfrac{1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s_0))\cdots)}{1-g_{\xi_{n-1}}(g_{\xi_{n-2}}(\cdots(g_{\xi_0}(s_0))\cdots)}}{\dfrac{1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_1}(s_0))\cdots)}{1-g_{\xi_{n-1}}(g_{\xi_{n-2}}(\cdots(g_{\xi_1}(s_0))\cdots)}} \, \frac{1-g_{\xi_{n-1}}(g_{\xi_{n-2}}(\cdots(g_{\xi_0}(s_0))\cdots)}{1-g_{\xi_{n-1}}(g_{\xi_{n-2}}(\cdots(g_{\xi_1}(s_0))\cdots)}. \tag{12}$$

Also, by part (4) of Lemma 2.1, $(1-g_{\xi_n}(t))/(1-t)$ decreases as t increases, and by parts (1) and (2) of Lemma 2.1,

$$g_{\xi_{n-1}}(g_{\xi_{n-2}}(\cdots(g_{\xi_0}(s_0))\cdots) \geq g_{\xi_{n-1}}(g_{\xi_{n-2}}(\cdots(g_{\xi_1}(s_0))\cdots).$$

These facts and (12) together imply that $(1-a_{n+1}(\bar{\xi}))/(1-a_n(T\bar{\xi}))$ decreases as n increases to some limit which we denote by $d(\bar{\xi})$. Hence

$$\lim_{n\to\infty} \log a_{n+1}(\bar{\xi})/\log a_n(T\bar{\xi}) = d(\bar{\xi}).$$

Since

$$\frac{1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s_0))\cdots)}{1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_1}(s_0))\cdots)} \leq 1 \tag{13}$$

for all n, $d(\bar{\xi})\leq 1$. It remains to prove that $d(\bar{\xi})>0$ w.p.1. We first note that by part (4) of Lemma 2.1,

$$1 \geq 1-g_{\xi_0}(s_0) \geq \frac{1-s_0}{m(\xi_0)}. \tag{14}$$

Therefore

$$-\log(1-g_{\xi_0}(s_0)) \leq -\log(1-s_0)+\log m(\xi_0) \tag{15}$$

and so

$$|\log(1-g_{\xi_0}(s_0))| \leq |\log(1-s_0)|+|\log m(\xi_0)|. \tag{16}$$

Taking expectations of (16) yields

$$\mathbf{E}|\log(1-g_{\xi_0}(s_0))| \leq |\log(1-s_0)|+\mathbf{E}|\log m(\xi_0)|<\infty. \tag{17}$$

Similarly, since

$$\frac{1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s_0))\cdots)}{1-s_0} \geq \frac{1}{\prod_{i=0}^{n} m(\xi_i)} \tag{18}$$

it follows that

$$\mathbf{E}|\log(1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s_0))\cdots)| < \infty. \tag{19}$$

For $n\geq 0$, let

$$f_n(\bar{\xi}) = \left|\log\frac{(1-a_{n+1}(\bar{\xi}))}{(1-a_n(T\bar{\xi})}\right|$$
$$= -\log\left(\frac{1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s_0))\cdots)}{1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_1}(s_0))\cdots)}\right). \tag{20}$$

Now, using (19), we can write

$$\mathbf{E}(f_n(\bar{\xi})) = \mathbf{E}(\log(1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_1}(s_0))\cdots))$$
$$-\mathbf{E}(\log(1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s_0))\cdots)). \tag{21}$$

But by the stationarity of $\bar{\xi}$,

$$\mathbf{E}(\log(1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_1}(s_0))\cdots))=$$
$$= \mathbf{E}(\log(1-g_{\xi_{-1}}(g_{\xi_{-2}}(\cdots(g_{\xi_{-n}}(s_0))\cdots)) \tag{22}$$

and

$$\mathbf{E}(\log(1-g_{\xi_n}(g_{\xi_{n-1}}(\cdots(g_{\xi_0}(s_0))\cdots))=$$
$$=\mathbf{E}(\log(1-g_{\xi_0}(g_{\xi_{-1}}(\cdots(g_{\xi_{-n}}(s_0))\cdots)). \tag{23}$$

Therefore

$$\mathbf{E}(f_n(\bar{\xi}))=-\mathbf{E}\left(\log\frac{(1-g_{\xi_0}(g_{\xi_{-1}}(\cdots(g_{\xi_{-n}}(s_0))\cdots))}{(1-g_{\xi_{-1}}(g_{\xi_{-2}}(\cdots(g_{\xi_{-n}}(s_0))\cdots))}\right). \tag{23}$$

But, by Lemma 2.3, the right-hand side of (23) converges w.p.1 to $\mathbf{E}(\log m(\xi_0))$ as n tends to infinity. Thus, by Fatou's Lemma,

$$0\leq\mathbf{E}\left(\underline{\lim}_{n\to\infty}f_n(\bar{\xi})\right)\leq\underline{\lim}_{n\to\infty}\mathbf{E}(f_n(\bar{\xi}))=\mathbf{E}(\log m(\xi_0))<\infty. \tag{24}$$

Hence $\underline{\lim}_{n\to\infty}f_n(\bar{\xi})<\infty$ w.p.1. But $\underline{\lim}_{n\to\infty}f_n(\bar{\xi})=|\log d(\bar{\xi})|$ (where we interpret $\log d(\bar{\xi})=+\infty$ if $d(\bar{\xi})=0$). Thus $d(\bar{\xi})>0$ w.p.1, and so the lemma is proved.

The following theorem is the analogue for B.P.R.E. of Heyde's exponential martingale theorem ([5]).

Theorem 3. *Let $\{Z_n\}_{n=0}^{\infty}$ be an increasing B.P.R.E. Let $Y_n=a_n(\bar{\xi})^{Z_n}$, $n\geq 0$, and denote by $F_n(\bar{\xi})$ the σ-field generated by $Z_0, Z_1,\ldots,Z_n$ and $\bar{\xi}$. Then $\{Y_n, F_n(\bar{\xi})\}_{n=0}^{\infty}$ is a positive bounded martingale and $\lim_{n\to\infty}Y_n=Y$ w.p.1 where Y is a random variable satisfying for almost all $\bar{\xi}$,*

(i) $\mathbf{P}(Y=0\mid\bar{\xi})=0$,
(ii) $\mathbf{P}(Y=1\mid\bar{\xi})=0$, *and*
(iii) $\mathbf{E}(Y\mid\bar{\xi})=s_0$.

Proof. It is easily checked that $\{Y_n, F_n(\bar{\xi})\}_{n=0}^{\infty}$ is a positive bounded martingale, and so, by the Martingale Convergence Theorem, $\lim_{n\to\infty}Y_n=Y$ w.p.1. Furthermore, for $u>0$, $\lim_{n\to\infty}\mathbf{E}(Y_n^u\mid\bar{\xi})=\mathbf{E}(Y^u\mid\bar{\xi})$ w.p.1. By the martingale property,

$$\mathbf{E}(Y_n\mid\bar{\xi})=\mathbf{E}(Y_0\mid\bar{\xi})=s_0 \tag{25}$$

(since $Z_0\equiv 1$) so that $\mathbf{E}(Y_n\mid\bar{\xi})=\lim_{n\to\infty}\mathbf{E}(Y_n\mid\bar{\xi})=s_0$ and so (iii) is proved.

To prove (i) and (ii), we follow a line of exposition similar to that of Athreya–Ney [2]. Let $\chi_n(u;\bar{\xi})=\mathbf{E}(Y_n^u\mid\bar{\xi})$ and $\chi(u;\bar{\xi})=\mathbf{E}(Y^u\mid\bar{\xi})$. Then one readily obtains

$$f_{\xi_1}(f_{\xi_2}(\cdots(f_{\xi_{n-1}}(e^{u\log a_n(\bar{\xi})}))\cdots)=\chi_{n-1}\left(\frac{u\log a_n(\bar{\xi})}{\log a_{n-1}(T\bar{\xi})};T\bar{\xi}\right). \tag{26}$$

(26) and Lemma 2.4 yield

$$\lim_{n\to\infty}\chi_{n-1}\left(\frac{u\log a_n(\bar{\xi})}{\log a_{n-1}(T\bar{\xi})};T\bar{\xi}\right)=\chi(ud(\bar{\xi});T\bar{\xi}). \tag{27}$$

From (26) and (27) we obtain the functional relation

$$\chi(u;\bar{\xi}) = f_{\xi_0}(\chi(ud(\bar{\xi}); T\bar{\xi})) \tag{28}$$

which is an extension to B.P.R.E. of the Heyde–Senata functional equation [5]. Now it is easily seen that

$$\lim_{u\downarrow 0} \chi(u;\bar{\xi}) = \mathbf{P}(Y \neq 0 \mid \bar{\xi}), \tag{29}$$

$$\lim_{u\uparrow \infty} \chi(u;\bar{\xi}) = \mathbf{P}(Y = 1 \mid \bar{\xi}). \tag{30}$$

By Lemma 2.4, $0 < d(\bar{\xi}) \leq 1$ w.p.1, so that $ud(\bar{\xi})$ decreases to zero as u decreases to zero and $ud(\bar{\xi})$ increases to infinity as u increases to infinity. Hence (29) and (30) imply that

$$\lim_{u\downarrow 0} \chi(ud(\bar{\xi}); T\bar{\xi}) = \mathbf{P}(Y \neq 0 \mid T\bar{\xi}), \tag{31}$$

$$\lim_{u\uparrow \infty} \chi(ud(\bar{\xi}); T\bar{\xi}) = \mathbf{P}(Y = 1 \mid T\bar{\xi}). \tag{32}$$

Using the functional relation (28) with equations (29) to (32) yields

$$\mathbf{P}(Y \neq 0 \mid \bar{\xi}) = f_{\xi_0}(\mathbf{P}(Y \neq 0 \mid T\bar{\xi})), \tag{33}$$

$$\mathbf{P}(Y = 1 \mid \bar{\xi}) = f_{\xi_0}(\mathbf{P}(Y = 1 \mid T\bar{\xi})). \tag{34}$$

We note that $\mathbf{E}(Y \mid \bar{\xi}) = s_0 < 1$ implies that $\mathbf{P}(Y = 1 \mid \bar{\xi}) < 1$ w.p.1. Applying Theorem 6 of [1] (and observing that the extinction probility is zero in our case) yields $\mathbf{P}(Y = 1 \mid \bar{\xi}) = 0$ w.p.1, so that part (ii) of the theorem is proved. It remains to prove part (i).

Let $0 < t < 1$. Then, since $\bar{\xi}$ is stationary,

$$\begin{aligned}\mathbf{P}(\mathbf{P}(Y \neq 0 \mid \bar{\xi}) \leq t) &= \mathbf{P}(\mathbf{P}(Y \neq 0 \mid T^n\bar{\xi}) \leq t) \\ &\leq \mathbf{P}(\mathbf{P}(Y \neq 0 \mid \bar{\xi}) \leq f_{\xi_0}(f_{\xi_1}(\cdots(f_{\xi_{n-1}}(t))\cdots)),\end{aligned} \tag{35}$$

where the last inequality follows from the fact that

$$\mathbf{P}(Y \neq 0 \mid \bar{\xi}) = f_{\xi_0}(f_{\xi_1}(\cdots(f_{\xi_{n-1}}(\mathbf{P}(Y \neq 0 \mid T^n\bar{\xi}))\cdots) \tag{36}$$

(which is obtained by iterating $(33)\cdots$). Letting n tend to infinity in (35) yields

$$\mathbf{P}(\mathbf{P}(Y \neq 0 \mid \bar{\xi}) \leq t) \leq \mathbf{P}(\mathbf{P}(Y \neq 0 \mid \bar{\xi}) = 0), \tag{37}$$

for $0 < t < 1$. Hence $\mathbf{P}(Y \neq 0 \mid \bar{\xi})$ equals 0 or 1 w.p.1. Since $\mathbf{E}(Y \mid \bar{\xi}) = s_0 \neq 0$, $\mathbf{P}(Y \neq 0 \mid \bar{\xi}) = 1$ w.p.1, and this concludes the proof of the theorem.

Theorem 4. *Let $\{Z_n\}_{n=0}^{\infty}$ be an increasig B.P.R.E. with $\mathbf{E}|\log m(\xi_0)| < \infty$. Then there exist normalizing constants $c_n(\bar{\xi})$, $n \geq 0$, such that $\lim_{n\to\infty} c_n^{-1} Z_n = W$ w.p.1, where*

(i) $\mathbf{P}(W=0\,|\,\bar{\xi})=0$ *w.p.*1, *and*
(ii) $\mathbf{P}(W<\infty\,|\,\bar{\xi})=1$ *w.p.*1.
Furthermore, $\lim_{n\to\infty} 1/n \log c_n = \mathbf{E}(\log m(\xi_0))$.

Proof. Let Y_n, $n \geq 0$, and Y be defined as in Theorem 3. For $n \geq 0$, let $c_n(\bar{\xi}) = (-\log a_n(\bar{\xi}))^{-1}$. Then, by Theorem 3, $\lim_{n\to\infty} c_n^{-1} Z_n = W$ w.p.1, where $W = |\log Y|$. Also,

$$\mathbf{P}(W=0\,|\,\bar{\xi}) = \mathbf{P}(Y=1\,|\,\bar{\xi}) = 0 \text{ w.p.1}, \tag{38}$$

and

$$\mathbf{P}(W=+\infty\,|\,\bar{\xi}) = \mathbf{P}(Y=0\,|\,\bar{\xi}) = 0 \text{ w.p.1}.$$

Furthermore since $\lim_{n\to\infty} 1/n \log W = 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log c_n(\bar{\xi}) = \lim_{n\to\infty} \frac{1}{n} \log Z_n = \mathbf{E}(\log m(\xi)) \text{ w.p.1}, \tag{39}$$

where the last equality is a consequence of Theorem 5.5 of [9].

3. Normalizing constants for B.P.R.E

Theorem 2. *Let* $\{Z_n\}_{n=0}^{\infty}$ *be a B.P.R.E. for which* $\mathbf{E}(\log m(\xi_0))>0$. *Let* $A = \{\omega: Z_n(\omega)\to+\infty \text{ as } n\to\infty\}$. *Then, for almost all* $\bar{\xi}$,

$$\mathbf{P}\left(A, \lim_{n\to\infty} \frac{W_n}{(1-q(T^n\bar{\xi}))Z_n} = 1\,\Big|\,\bar{\xi}\right) = \mathbf{P}(A\,|\,\bar{\xi}) = 1-q(\bar{\xi}), \tag{40}$$

where $\{W_n\}_{n=0}^{\infty}$ is the reduced branching process.

Proof. Let $\varepsilon>0$ be arbitrary. Then, by a conditional form of Chebyshev's Inequality,

$$\mathbf{P}\left(\left|\frac{W_n}{(1-q(T^n\bar{\xi}))Z_n}-1\right|>\varepsilon\,\Big|\,Z_n,\bar{\xi}\right) \leq \frac{\mathbf{E}((W_n-Z_n(1-q(T^n\bar{\xi})))^2\,|\,Z_n,\bar{\xi})}{\varepsilon^2 Z_n^2(1-q(T^n\bar{\xi}))^2}. \tag{41}$$

Now, conditioned in Z_n and $\bar{\xi}$, W_n is distributed as the number of successes in Z_n independent trials, where the probability of success is $(1-q(T^n\bar{\xi}))$. Thus (41) becomes

$$\mathbf{P}\left(\left|\frac{W_n}{(1-q(T^n\bar{\xi}))Z_n}-1\right|>\varepsilon\,\Big|\,Z_n,\bar{\xi}\right) \leq \frac{Z_n(1-q(T^n\bar{\xi}))q(T^n\bar{\xi})}{\varepsilon^2 Z_n^2(1-q(T^n\bar{\xi}))^2} \leq \frac{1}{\varepsilon^2 Z_n(1-q(T^n\bar{\xi}))}. \tag{42}$$

It is shown in the proof of Theorem 5.7 of [9] that $\lim_{n\to\infty} 1/n \log(1-q(T^n\bar{\xi}))=0$ w.p.1. Furthermore, by the Classification Theorem (Thm. 5.5 of [9]), $\lim_{n\to\infty} 1/n \log Z_n = \mathbf{E}(\log m(\xi_0))$ almost everywhere in A. Thus, almost everywhere in A,

$$\sum_{n=1}^{\infty} \mathbf{P}\left(\left|\frac{W_n}{(1-q(T^n\bar{\xi}))Z_n}-1\right| > \varepsilon \,\Big|\, Z_n, \bar{\xi}\right) < \infty. \tag{43}$$

By Levy's form of the Borel–Cantelli Lemma, it follows, that, for almost all $\bar{\xi}$,

$$\mathbf{P}\left(A, \left|\frac{W_n}{(1-q(T^n\bar{\xi}))Z_n}-1\right| > \varepsilon \text{ i.o.} \,\Big|\, \bar{\xi}\right) = 0 \tag{44}$$

from which the theorem is a direct consequence.

Proof of Theorem 1. If $\mathbf{E}(\log m(\xi_0)) \leq 0$, then it follows from the Classification Theorem (Theorem 5.5 of [9]) that $c_n(\bar{\xi}) = \prod_{i=0}^{n-1} m(\xi_i)$ will satisfy the theorem. Hence we need only consider the case $\mathbf{E}(\log m(\xi_0)) > 0$. Also we may assume $\mathbf{P}(q(\bar{\xi})<1)=1$, for otherwise $\{Z_n\}_{n=0}^{\infty}$ becomes extinct w.p.1, and the preceeding constants $\{c_n\}_{n=1}^{\infty}$ would suffice.

Consider the associate B.P.R.E. $\{\hat{Z}_n\}$ (see [9], Section 3). Then $\{\hat{Z}_n\}_{n=0}^{\infty}$ is an increasing B.P.R.E. and $\mathbf{E}(\log \hat{m}(\xi_0)) = \mathbf{E}(\log m(\xi_0)) > 0$ ([9], Thm. 5.7). Hence by Theorem 4, there exist normalizing constants $\hat{c}_n(\bar{\xi})$, $n \geq 0$, such that $\lim_{n\to\infty} \hat{c}_n^{-1}\hat{Z}_n = \hat{W}$ w.p.1 where $\hat{W}$ is a random variable satisfying $\mathbf{P}(0 < W < +\infty \mid \bar{\xi}) = 1$ w.p.1. Also,

$$\lim_{n\to\infty} \frac{1}{n} \log \hat{c}_n(\bar{\xi}) = \mathbf{E}(\log \hat{m}(\xi_0)) = \mathbf{E}(\log m(\xi_0)). \tag{45}$$

Let $\{W_n\}_{n=0}^{\infty}$ be the reduced branching process and $A = \{\omega: Z_n(\omega) \to +\infty$ as $n\to\infty\}$. Then, by Theorem 4.5 of [9], $\lim_{n\to\infty} \hat{c}_n^{-1} W_n = \hat{W}$ almost everywhere on A. Then Theorem 2 implies that $\lim_{n\to\infty} \hat{c}_n^{-1}(1-q(T^n\bar{\xi}))Z_n = \hat{W}$ almost everywhere on A. Let $c_n(\bar{\xi}) = \hat{c}_n(1-q(T^n\bar{\xi}))^{-1}$ for $n \geq 0$ and define a new random variable W by $W(\omega) = \hat{W}(\omega)$ for $w \in A$, and W is zero elsewhere. Then it follows directly that $\lim c_n^{-1} Z_n = W$ w.p.1, where

$$\mathbf{P}(W < +\infty \mid \bar{\xi}) = \mathbf{P}(\hat{W} < \infty \mid \bar{\xi}) = 1 \text{ w.p.1},$$

$$\mathbf{P}(W = 0 \mid \bar{\xi}) = \mathbf{P}(A^c \mid \bar{\xi}) = q(\bar{\xi}) \text{ w.p.1}.$$

Furthermore,

$$\frac{1}{n}\log c_n(\bar{\xi}) = \frac{1}{n}\log \hat{c}_n(\xi) - \frac{1}{n}\log(1-q(T^n\bar{\xi})). \tag{46}$$

As was previously noted, $1/n \log(1-q(T^n\bar{\xi}))$ converges to zero w.p.1, and $\lim_{n\to\infty} 1/n \log \hat{c}_n(\bar{\xi}) = \mathbf{E}(\log m(\xi_0))$ w.p.1 by (45), so that $\lim_{n\to\infty} 1/n \log c_n(\bar{\xi}) = \mathbf{E}(\log m(\xi_0))$ w.p.1.

It remains to prove that $c_{n+1}(\bar{\xi})/c_n(\bar{\xi})$ converges in distribution to $m(\xi_0)$. Now

$$\frac{c_{n+1}(\bar{\xi})}{c_n(\bar{\xi})}=\frac{(1-q(T^n\bar{\xi}))}{(1-q(T^{n+1}\bar{\xi}))}\cdot\frac{\hat{c}_{n+1}(\bar{\xi})}{\hat{c}_n(\bar{\xi})}$$
$$=\frac{(1-q(T^n\bar{\xi}))}{(1-q(T^{n+1}\bar{\xi}))}\cdot\frac{\log\hat{g}_{\xi_{n-1}}(\hat{g}_{\xi_{n-2}}(\cdots(\hat{g}_{\xi_0}(s_0))\cdots)}{\log\hat{g}_{\xi_n}(\hat{g}_{\xi_{n-1}}(\cdots(\hat{g}_{\xi_0}(s_0))\cdots)} \tag{47}$$

where g_{ξ_i} is the probability generating function of $\hat{\xi}_i$. But, by the stationarity of $\bar{\xi}$, the right-hand side of (47) is equal in distribution to

$$\frac{(1-q(\bar{\xi}))}{(1-q(T\bar{\xi}))}\cdot\frac{\log\ \hat{g}_{\xi_{-1}}(\hat{g}_{\xi_{-2}}(\cdots(\hat{g}_{\xi_{-n}}(s_0))\cdots)}{\log\hat{g}_{\xi_0}(\hat{g}_{\xi_{-1}}(\cdots(\hat{g}_{\xi_{-n}}(s_0))\cdots)}.$$

Furthermore, following the line of argument used in Lemma 2.4 we get

$$\lim_{n\to\infty}\frac{\log\hat{g}_{\xi_0}(\hat{g}_{\xi_{-1}}(\cdots(\hat{g}_{\xi_{-n}}(s_0))\cdots)}{\log\hat{g}_{\xi_{-1}}(\hat{g}_{\xi_{-2}}(\cdots(\hat{g}_{\xi_{-n}}(s_0))\cdots)}$$
$$=\lim_{n\to\infty}\frac{1-\hat{g}_{\xi_0}(\hat{g}_{\xi_{-1}}(\cdots(\hat{g}_{\xi_{-n}}(s_0))\cdots)}{1-\hat{g}_{\xi_{-1}}(\hat{g}_{\xi_{-2}}(\cdots(\hat{g}_{\xi_{-n}}(s_0))\cdots)}=\frac{1}{\hat{m}(\xi_0)}\text{ w.p.1} \tag{48}$$

where the last equality is from (6) (as applied to the $\{{}^nZ_n\}_{n=0}^{\infty}$ process). Hence

$$\lim_{n\to\infty}\frac{(1-q(\bar{\xi}))}{(1-q(T\bar{\xi}))}\cdot\frac{\log g_{\xi_{-1}}(g_{\xi_{-2}}(\cdots(g_{\xi_{-n}}(s_0))\cdots)}{\log g_{\xi_0}(g_{\xi_{-1}}(\cdots(g_{\xi_{-n}}(s_0))\cdots)}$$
$$=\frac{(1-q(\bar{\xi}))}{(1-q(T\bar{\xi}))}\hat{m}(\xi_0)\text{ w.p.1}=m(\xi_0) \tag{49}$$

where the last equality is from equation (24) of [9]. It now follows that $c_{n+1}(\bar{\xi})/c_n(\bar{\xi})$ converges in distribution to $m(\xi_0)$ and so the theorem is proved.

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